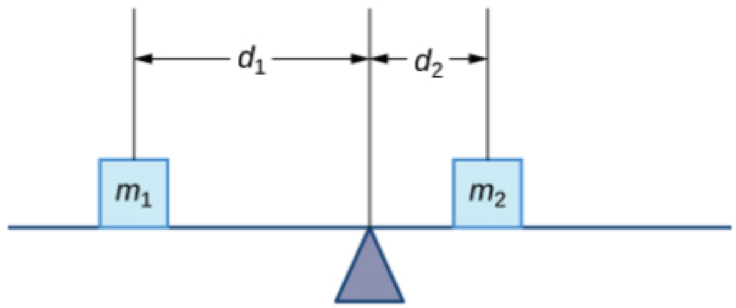
SECTION 16.6: MASS CALCULATIONS

ONE DIMENSIONAL SYSTEMS:

DISCRETE SYSTEMS:

BIG IDEA: Suppose we put two masses m_1 and m_2 on a board as pictured below and wish to place a fulcrum between the masses so as to balance the board.



For the board above to balance at the fulcrum, we need that $m_1 d_1 = m_2 d_2$. This is called the **principle of the lever**. To help us find where to place the fulcrum, we model the scenario more precisely. We imagine the board as the x-axis with a mass of m_1 located at the point x_1 and a mass of m_2 located at the point x_2. Our goal is to find $\overline{x}$, the so-called **center of mass** of the system. In many situations, such as straight-line kinematics (momentum and kinetic energy), problems involving a system of masses can be reduced to a simpler problem involving the total mass of the system located at the center of mass.

Using the principle of the lever, we know that:

$$\begin{aligned}
m_1 d_1 &= m_2 d_2 \\
m_1|\overline{x} - x_1| &= m_2|\overline{x} - x_2| \\
m_1(\overline{x} - x_1) &= m_2(x_2 - \overline{x}) \\
m_1\overline{x} - m_1 x_1 &= m_2 x_2 - m_2\overline{x} \\
m_1\overline{x} + m_2\overline{x} &= m_1 x_1 + m_2 x_2 \\
\overline{x}(m_1 + m_2) &= m_1 x_1 + m_2 x_2 \\
\overline{x} &= \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}
\end{aligned}$$

The quantity $m = m_1 + m_2$ is the **total mass** of the system. The quantity $M = m_1 x_1 + m_2 x_2$ is called the **moment of the system about the origin**. In general, a moment is a mass times a displacement. So in this case, $m_1 x_1$ is the moment of m_1 about the origin since we are multiplying m_1 about its displacement from $x = 0$.

In general, if we have n masses $m_1, m_2, \ldots m_n$ located at $x_1, x_2, \ldots x_n$, then:

$$m = \sum_{k=1}^{n} m_k, \quad M = \sum_{k=1}^{n} m_k x_k, \quad \overline{x} = \frac{M}{m}$$

EXAMPLE 1: Consider the system of three masses: $m_1 = 10$ at $x_1 = -3$, $m_2 = 5$ at $x_2 = 1$, and $m_3 = 2$ at $x_3 = 4$. Find the center of mass, $\overline{x}$.

Ans: $m = 17$, $M = -17$ so $\overline{x} = -1$.

CONTINUOUS SYSTEMS: Suppose we model a wire as an interval $[a, b]$ and let $\rho(x)$ be the **linear density** of the wire. This means that $\rho(x)$ tells you the density of the wire, with units of mass per length, at location x.

Using the old 'chop and add' approach to integration, we can show that:

$$m = \int_a^b \rho(x)\, dx, \quad M = \int_a^b x\, \rho(x)\, dx, \quad \overline{x} = \frac{M}{m}$$

EXAMPLE 2: A wire is modeled by $[0, 1]$ with density $\rho(x) = e^{-x}$. Find the center of mass of the wire.

Ans: $m = 1 - e^{-1} = \dfrac{e-1}{e}$, $M = 1 - 2e^{-1} = \dfrac{e-2}{e}$, so $\overline{x} = \dfrac{e-2}{e-1}$.

TWO DIMENSIONAL SYSTEMS:

DISCRETE SYSTEM:

Suppose we locate a mass m_1 at the point (x_1, y_1). Then we define:

- M_x is the **moment about the** x**-axis**.

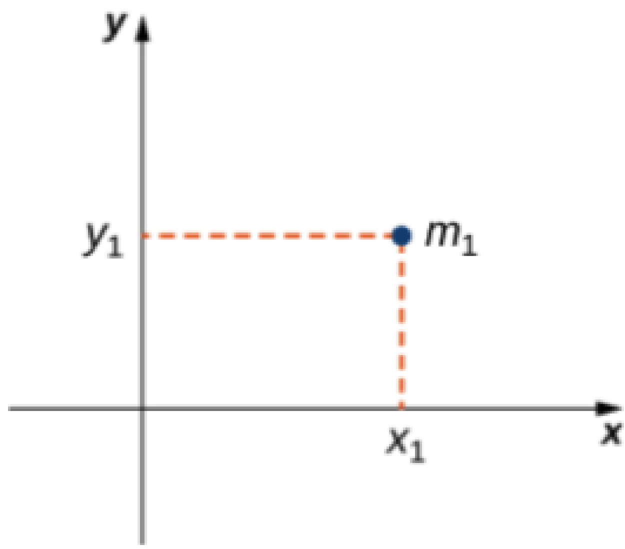
  The moment about the <u>x-axis</u> is the product of the mass m_1 with its displacement from the <u>x-axis</u>.

  Hence, $M_x = m_1 y_1$.

- M_y is the **moment about the** y**-axis**.

  The moment about the <u>y-axis</u> is the product of the mass m_1 with its displacement from the <u>y-axis</u>.

  Hence, $M_y = m_1 x_1$.



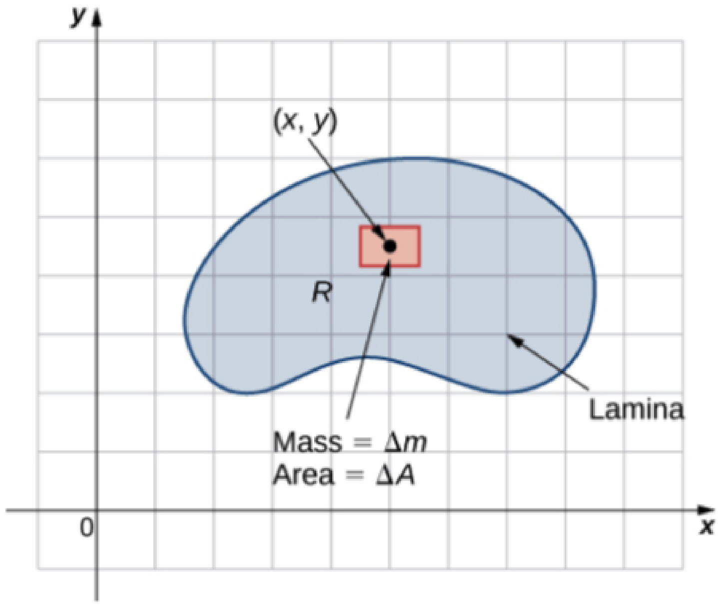
In general, if we have n masses m_1, m_2, $\ldots m_n$ located at (x_1, y_1), (x_2, y_2), $\ldots (x_n, y_n)$, then:

$$m = \sum_{k=1}^{n} m_k, \quad M_x = \sum_{k=1}^{n} m_k y_k, \quad M_y = \sum_{k=1}^{n} m_k x_k, \quad (\overline{x}, \overline{y}) = \left(\frac{M_y}{m}, \frac{M_x}{m} \right)$$

EXAMPLE 3: Consider the system of three masses: $m_1 = 10$ at $(x_1, y_1) = (-3, 2)$, $m_2 = 5$ at $(x_2, y_2) = (1, 3)$, and $m_3 = 2$ at $(x_3, y_3) = (4, -5)$. Find the center of mass, $(\overline{x}, \overline{y})$.

Ans: $m = 17$, $M_x = 25$, $M_y = -17$ so $(\overline{x}, \overline{y}) = \left(-1, \dfrac{25}{17}\right)$

CONTINUOUS SYSTEMS: Suppose we model planar lamina as a region R in the xy-plane and let $\rho(x,y)$ be the **planar density** of the region. This means that $\rho(x,y)$ tells you the density of the lamina, with units of mass per area, at location (x,y).



Using the old 'chop and add' approach to integration, we can show that:

$$m = \iint_R \rho(x,y)\,dA, \quad M_x = \iint_R y\rho(x,y)\,dA, \quad M_y = \iint_R x\rho(x,y)\,dA, \quad (\overline{x},\overline{y}) = \left(\frac{M_y}{m}, \frac{M_x}{m}\right)$$

EXAMPLE 4: Suppose a planar lamina is modeled by the region R in Quadrant I bounded by $y = \sqrt{4x - x^2}$.

If the planar density of R is given by $\rho(x,y) = xy$, find the center of mass $(\overline{x},\overline{y})$ of R.

Ans: Using polar coordinates: $m = \dfrac{32}{3}$, $M_x = 4\pi$, $M_y = \dfrac{128}{5}$ so $(\overline{x},\overline{y}) = \left(\dfrac{12}{5}, \dfrac{3\pi}{8}\right)$

DEFINITION: The **centroid** of a planar region is the center of mass of the region assuming a density $\rho(x, y) = 1$.

EXAMPLE 5: Find the centroid of the triangle with vertices $(0,0)$, $(b,0)$ and $(0, h)$.

Ans: $m = \text{area} = \dfrac{1}{2}bh,\ M_y = \dfrac{1}{6}b^2h,\ M_x = \dfrac{1}{6}bh^2$ so $(\overline{x}, \overline{y}) = \left(\dfrac{1}{3}b, \dfrac{1}{3}h\right)$

THREE DIMENSIONAL SYSTEMS: Suppose we locate a mass m_1 at the point (x_1, y_1, z_1). Then we define:

- M_{xy} is the **moment about the** xy**-plane**.

  The moment about the $\underline{xy\text{-plane}}$ is the product of the mass m_1 with its displacement from the $\underline{xy\text{-plane}}$.

  Hence, $M_{xy} = m_1 z_1$.

- M_{xz} is the **moment about the** xz**-plane**.

  The moment about the $\underline{xz\text{-plane}}$ is the product of the mass m_1 with its displacement from the $\underline{xz\text{-plane}}$.

  Hence, $M_{xz} = m_1 y_1$.

- M_{yz} is the **moment about the** yz**-plane**.

  The moment about the $\underline{yz\text{-plane}}$ is the product of the mass m_1 with its displacement from the $\underline{yz\text{-plane}}$.

  Hence, $M_{yz} = m_1 x_1$.

DISCRETE SYSTEMS:

Suppose we have n masses m_1, m_2, $\ldots m_n$ located at (x_1, y_1, z_1), (x_2, y_2, z_2), $\ldots (x_n, y_n, z_n)$.

Generalize the formulas from one and two dimensions to write equations for the following quantities:

- $m =$

- $M_{xy} =$

- $M_{xz} =$

- $M_{yz} =$

- $(\overline{x}, \overline{y}, \overline{z}) =$

CONTINUOUS SYSTEMS: Suppose a region Q in space has a density function $\rho(x, y, z)$. This means that $\rho(x, y, z)$ tells you the density of the solid, with units of mass per volume, at location (x, y, z). Generalize the formulas from one and two dimensions to write equations for the following quantities:

- $m =$

- $M_{xy} =$

- $M_{xz} =$

- $M_{yz} =$

- $(\overline{x}, \overline{y}, \overline{z}) =$

EXAMPLE 6: Find the centroid of the solid bounded by $z = \sqrt{18 - x^2 - y^2}$ and $z = \sqrt{x^2 + y^2}$.

HINT: Use symmetry to simplify some of the calculations.

Ans: $m = 36\pi(\sqrt{2} - 1)$, $M_{xz} = M_{yz} = 0$, $M_{xy} = \dfrac{81\pi}{2}$ so $(\overline{x}, \overline{y}, \overline{z}) = \left(0, 0, \dfrac{8}{9}\left(\sqrt{2} - 1\right)\right)$

SECOND MOMENTS: MOMENTS OF INERTIA

The moments we have studied so far are sometimes called **first** moments because they involve multiplying the mass by the **first** power of displacement. Moments of **inertia** are computed by multiplying mass by the **square** of the displacement. The moment of inertia about the origin, I_o for the system below is: $I_o = m_1 x_1^2 + m_2 x_2^2$.

The analog of 'center of mass' for moments of inertia is the **radius of gyration**, which here is $\overline{r_o} = \sqrt{\dfrac{I}{m}}$.

What $\overline{x}$ does for straight-line motion $\overline{r_o}$ does for rotational motion. We can repeat the process for second moments that we did for first moments to obtain the following formulas.

ONE DIMENSIONAL SYSTEMS: In one dimension, we only have a moment of inertia about the origin.

DISCRETE: If we have n masses $m_1, m_2, \ldots m_n$ located at $x_1, x_2, \ldots x_n$, then:

$$I_o = \sum_{k=1}^{n} m_k x_k^2, \quad \overline{r_o} = \sqrt{\frac{I}{m}}$$

CONTINUOUS: If we model a wire as an interval $[a, b]$ and let $\rho(x)$ be the linear density of the wire. Then:

$$I_o = \int_a^b x^2 \rho(x)\, dx, \quad \overline{r_o} = \sqrt{\frac{I}{m}}$$

TWO DIMENSIONAL SYSTEMS: We have moments of inertia about the x-axis, y-axis, and origin.

DISCRETE: If we have n masses $m_1, m_2, \ldots m_n$ located at $(x_1, y_1), (x_2, y_2), \ldots (x_n, y_n)$, then:

$$I_x = \sum_{k=1}^{n} m_k y_k^2, \quad I_y = \sum_{k=1}^{n} m_k x_k^2, \quad I_o = \sum_{k=1}^{n} m_k (x_k^2 + y_k^2) = I_x + I_y, \quad \overline{r_x} = \sqrt{\frac{I_x}{m}}, \quad \overline{r_y} = \sqrt{\frac{I_y}{m}}, \quad \overline{r_o} = \sqrt{\frac{I_o}{m}}$$

CONTINUOUS: If a planar lamina is modeled by a region R in the xy-plane and with density $\rho(x, y)$, then:

$$I_x = \iint_R y^2 \rho(x,y)\, dA, \quad I_y = \iint_R x^2 \rho(x,y)\, dA, \quad I_o = \iint_R (x^2 + y^2)\rho(x,y)\, dA = I_x + I_y$$

$$\overline{r_x} = \sqrt{\frac{I_x}{m}}, \quad \overline{r_y} = \sqrt{\frac{I_y}{m}}, \quad \overline{r_o} = \sqrt{\frac{I_o}{m}}$$

EXAMPLE 7: Suppose a solid is modeled by a region Q in space with density $\rho(x, y, z)$. Let I_x denote the moment of inertia of Q about the x-axis. Find an integral formula for I_x.

HINT: Given a point (x, y, z), the distance^2 to the x-axis is: $y^2 + z^2 \ldots$

HOMEWORK: Section 16.6: 7 - 63 every other odd.